

Faster 3-Periodic Merging Networks

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Abstract. We consider the problem of merging two sorted sequences on a comparator network that is used repeatedly, that is, if the output is not sorted, the network is applied again using the output as input. The challenging task is to construct such networks of small depth. The first constructions of merging networks with a constant period were given by Kutyłowski, Loryś and Oesterdikhoff [7]. They have given 3-periodic network that merges two sorted sequences of N numbers in time $12\log N$ and a similar network of period 4 that works in $5.67\log N$. We present a new family of such networks that are based on Canfield and Williamson periodic sorter [4]. Our 3-periodic merging networks work in time upper-bounded by $6\log N$. The construction can be easily generalized to larger constant periods with decreasing running time, for example, to 4-periodic ones that work in time upper-bounded by $4\log N$. Moreover, to obtain the facts we have introduced a new proof technique.

Keywords: parallel merging, comparison networks, merging networks, periodic networks, comparators, oblivious merging.

1 Introduction

Comparator networks are probably the simplest parallel model that is used to solve such tasks as sorting, merging or selecting [6]. Each network represents a data-oblivious algorithm, which can be easily implemented in hardware. Moreover, sorting networks can be applied in secure, multi-party computation (SMC) protocols. They are also strongly connected with switching networks [9]. The most famous constructions of sorting networks are Odd-Even and Bitonic networks of depth $\frac{1}{2}\log^2 N$ due to Batcher [2] and AKS networks of depth $O(\log N)$ due to Ajtai, Komlos and Szemerédi [1]. The long-standing disability to decrease a large constant hidden behind the asymptotically optimal complexity of AKS networks to a practical value [15] has resulted in studying easier, sorting-related problems, whose optimal networks have small constants.

A comparator network consists of a set of N registers, each of which can contain an item from a totally ordered set, and a sequence of comparator stages. Each stage is a set of comparators that connect disjoint pairs of registers and, therefore, can work in parallel (a comparator is a simple device that takes a contents of two registers and performs a compare-exchange operation on them: the minimum is put into the first register and the maximum into the second one). Stages are run one after another in

synchronous manner, hence we can consider the number of stages as the running time. The size of a network is defined to be the total number of comparators in all its stages.

A network A consisting of stages S_1, S_2, \dots, S_d is called p -periodic if $p < d$ and for each i , $1 \leq i \leq d - p$, stages S_i and S_{i+p} are identical. A periodic network is easy to implement, especially in hardware, because one can use the first p stages in a cycle: if the output of p -th stage is not correct (sorted, for example), the sequence of p stages is run again. We can also define a p -periodic network just by giving the total number of stages and a description of its first p stages. A challenging task is to construct a family of small-periodic networks for sorting-related problems with the running time equal to, or not much greater than that of non-periodic networks.

Dowd et al. [5] gave the construction of $\log N$ -periodic sorting networks of N registers with running time of $\log^2 N$. Kutylowski et al. [8] introduced a general method to convert a non-periodic sorting network into a 5-periodic one, but the running time increases by a factor of $O(\log N)$ during the conversion. For simpler problems such as merging or correction there are constant-periodic networks that solve the corresponding problem in asymptotically optimal logarithmic time [7,13]. In particular, Kutylowski, Loryś and Oesterdikhoff [7] have given 3-periodic network that merges two sorted sequences of N numbers in time $12 \log N$ and a similar network of period 4 that works in $5.67 \log N$. They have also sketched a construction of merging networks with periods larger than 4 and running time decreasing asymptotically to $2.25 \log N$.

In this paper, we introduce a new family of constant-periodic merging networks that are based on the Canfield and Williamson $O(\log N)$ -periodic sorter [4] by a certain periodification technique. Our 3-periodic merging networks work in time upper-bounded by $6 \log N$ and 4-periodic ones - in time upper-bounded by $4 \log N$. The construction can be easily generalized to larger constant periods with decreasing running time.

The advantage of constant-periodic networks is that they have pretty simple patterns of communication links, that is, each node (register) of such a network can only be connected to a constant number of other nodes. Such patterns are easier to implement, for example, in hardware. Moreover, a node uses these links in a simple periodic manner and this can save control logic and simplify timing considerations.

2 Periodic merging networks

Our merging networks are based on the Canfield and Williamson [4] $O(\log N)$ -periodic sorters. We recall now the definition of their networks: for each $k \geq 1$ let $CW_k = S_1, \dots, S_k$ denote a network of $N = 2^k$ registers, where the stages are defined as follows (see also Figures 1 and 2):

$$S_1 = \left\{ [2i : 2i + 1] : i = 0, 1, \dots, 2^{k-1} - 1 \right\}, \quad (1)$$

$$S_{j+1} = \left\{ [2i + 1 : 2i + 2^{k-j}] : i = 0, 1, \dots, 2^{k-1} - 2^{k-j-1} - 1 \right\}, j = 1, \dots, k - 1. \quad (2)$$

The merging and sorting properties of the networks are given in the following proposition.

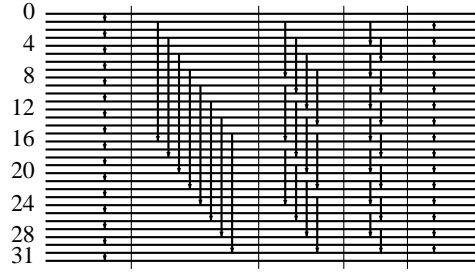


Fig. 1. The Canfield and Williamson $\log N$ -periodic sorter, where $N = 32$. Registers and comparators are represented by horizontal lines and arrows, respectively. Stages are separated by vertical lines.

Proposition 1. (1) For each $k \geq 1$, if two sorted sequences of length 2^{k-1} are given in registers with odd and even indices, respectively, then CW_k is a merging network. (2) For each $k \geq 1$, CW_k is a k -pass periodic sorting network.

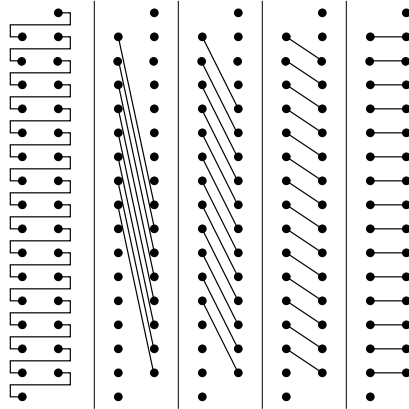


Fig. 2. Another view of CW_5 5-pass 5-periodic sorter. Registers and comparators are represented by dots and edges, respectively. Stages are separated by vertical lines.

We would like to implement a version of this network as a constant-periodic comparator network. Consider first the most challenging 3-periodic implementation. We start with the definition of a temporally construction P_k which structure is similar to the structure of CW_k . Then we transform it to 3-periodic network M_k . The idea is to replace each register i in CW_k (except the first and the last ones) with a sequence of $k - 2$ consecutive registers, move the endpoints of long comparators one register further or closer depending on the parity of i and insert between each pair of stages containing long com-

parators a stage with short comparators joining the endpoints of those long ones. The result is depicted in Fig. 3. In this way, we obtain a network in which each register is used in at most three consecutive stages. Therefore the network P_k can be packed into the first 3 stages and used periodically to get the desired 3-periodic merging network.

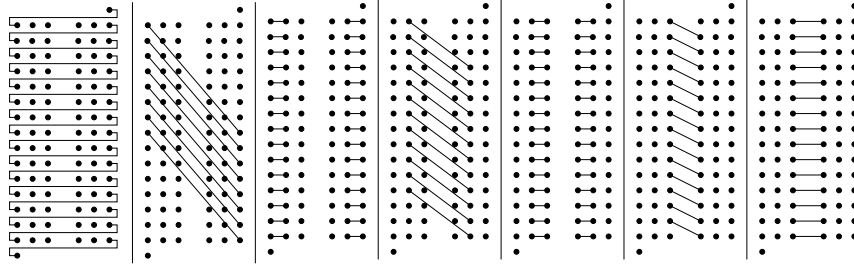


Fig. 3. P_5 as an implementation of CW_5 . Registers and comparators are represented by dots and edges, respectively. Stages are separated by vertical lines. Stages with short horizontal comparators are inserted between stages with long comparators.

Let $[i : j]$ denote a comparator connecting registers i and j . A comparator $[i : j]$ is *standard* if $i < j$. For an N -register network $A = S_1, S_2, \dots, S_d$, where S_1, S_2, \dots, S_d denote stages, and for an integer $j \in \{1, \dots, N\}$, we will use the following notations:

$$fst(j, A) = \min \{1 \leq i \leq d : j \in regs(S_i)\} \quad (3)$$

$$lst(j, A) = \max \{1 \leq i \leq d : j \in regs(S_i)\} \quad (4)$$

$$delay(A) = \max_{j \in \{1, \dots, N\}} \{lst(j, A) - fst(j, A) + 1\} \quad (5)$$

where $regs(\{[i_1 : j_1], \dots, [i_r : j_r]\})$ denotes the set $\{i_1, j_1, \dots, i_r, j_r\}$.

Let us define formally the new family of merging networks. For each $k \geq 3$ we would like to transform the network CW_k into a new network P_k .

Definition 1. Let $n_k = 2^{k-1} - 1$ be one less than the half of the number of registers in CW_k and $b_k = 2(k - 2)$. The number of registers of P_k is defined to be $N_k = n_k \cdot b_k + 2$. The stages of $P_k = S_{k,1} \cup \{[0 : 1], [N_k - 2 : N_k - 1]\}, S_{k,2}, \dots, S_{k,2k-3}$ are defined by the following equations, where $j = 1, \dots, \frac{b_k}{2}$:

$$S_{k,1} = \{[b_k i : b_k i + 1] : i = 1, \dots, n_k - 1\} \quad (6)$$

$$S_{k,2j} = \left\{ [b_k i + j : b_k(i + 2^{k-j-1} - 1) + (b_k - j + 1)] : i = 0, \dots, n_k - 2^{k-j-1} \right\} \quad (7)$$

$$S_{k,2j+1} = \{[b_k i + j : b_k i + j + 1], \quad (8)$$

$$[b_k i + (b_k - j) : b_k i + (b_k - j + 1)] : i = 0, \dots, n_k - 1\} \quad (9)$$

The network P_5 is depicted in Figure 4.

Fact 1 $delay(P_k) = 3$ for $k \geq 3$. □

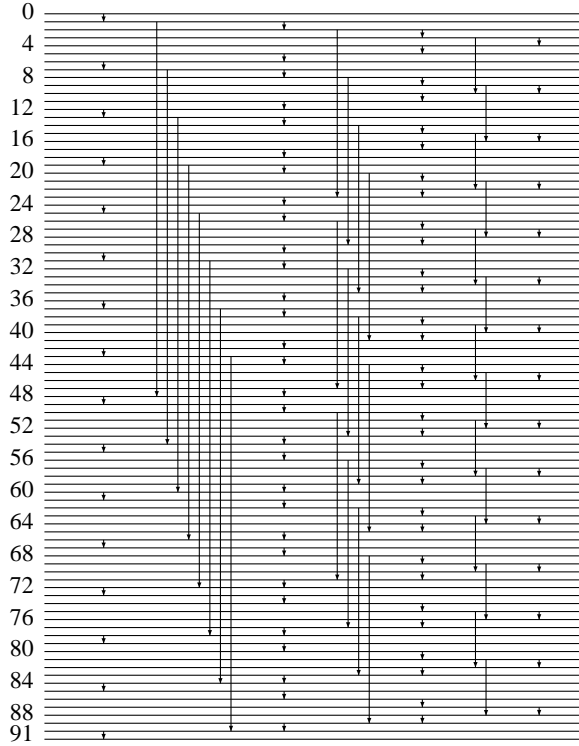


Fig. 4. The traditional drawing of P_5 network

Let $A = S_1, S_2, \dots, S_d$ and $A' = S'_1, S'_2, \dots, S'_{d'}$ be N -input comparator networks such that for each i , $1 \leq i \leq \min(d, d')$, $\text{regs}(S_i) \cap \text{regs}(S'_i) = \emptyset$. Then $A \cup A'$ is defined to be $(S_1 \cup S'_1), (S_2 \cup S'_2), \dots, (S_{\max(d, d')} \cup S'_{\max(d, d')})$, where empty stages are added at the end of the network of smaller depth.

For any comparator network $A = S_1, \dots, S_d$ and $D = \text{delay}(A)$, let us define a network $B = T_1, \dots, T_D$ to be a *compact form* of A , where $T_q = \bigcup \{S_{q+pD} : 0 \leq p \leq (d-q)/D\}$, $1 \leq q \leq D$. Observe that B is correctly defined due to the delay of A . Moreover, $\text{depth}(B) = \text{delay}(B) = \text{delay}(A)$.

Definition 2. For $k \geq 3$ let M_k denote the compact form of P_k with the first and the last registers deleted. That is, the network $M_k = T_1^k, T_2^k, T_3^k$ is using the set of registers numbered $\{1, 2, \dots, N_k\}$, where $N_k = (2^{k-1} - 1) \cdot 2(k-2)$, and $T_j^k = \{S_{k,j+3i} : 0 \leq i \leq \frac{2k-j-3}{3}\}$, $j = 1, 2, 3$.

It is not necessary to delete the first and the last registers of P_k but this will simplify proofs a little bit in the next section. The network M_5 is given in Fig. 5.

Theorem 2. There exists a family of 3-periodic comparator networks M_k , $k \geq 3$, such that each M_k is a $2k-5$ -pass merger of two sorted sequences given in odd and even

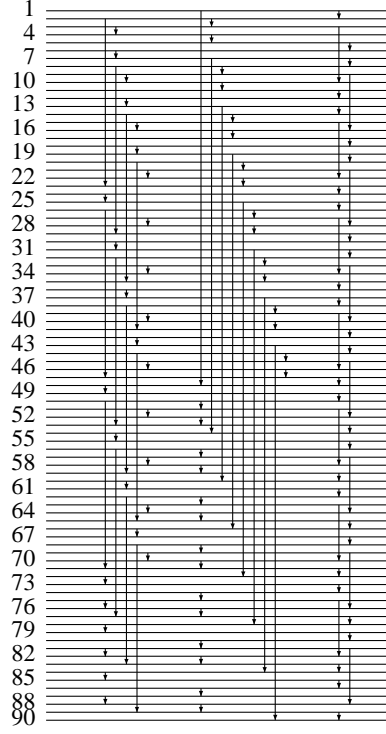


Fig. 5. The M_5 network

registers, respectively. The running time of M_k is $6k - 15 \leq 6\log N_k$, where $N_k = (2^k - 2)(k - 2)$ is the number of registers in M_k .

The proof is based on the observation that M_k merges $k - 2$ pairs of sorted subsequences, one after another, in pipeline fashion. Details are given in the next section.

In a similar way, we can convert CW_k into a 4-periodic merging network. Assume that k is even. We replace each register (except the first and the last ones) with a sequence of $(k - 2)/2$ consecutive registers, move the endpoints of long comparators in such a way that exactly two long comparators start or end at each new register and insert after each pair of stages containing long comparators a stage with short comparators joining the endpoints of those long comparators. The result is depicted in Fig. 6.

3 Proof of Theorem 2

The first observation we would like to make is that we can consider inputs consisting of 0's and 1's only. The well-known Zero-One Principle states that any comparator network that sorts 0-1 input sequences correctly sorts also arbitrary input sequences [6]. In the similar way, we can prove that the same property holds also for merging:

Proposition 2. *If a comparator network merges any two 0-1 sorted sequences, then it correctly merges any two sorted sequences.* \square

Therefore we can analyze computations of the network M_k , $k \geq 3$, by describing each state of registers as a 0-1 sequence $\bar{x} = (x_1, \dots, x_{N_k})$, where x_i represents the content of register i . If \bar{x} is an input sequence for $2k-5$ passes of M_k , then by $\bar{x}^{(i)}$ we denote the content of registers after i passes of M_k , $i = 0, \dots, 2k-5$, that is, $\bar{x}^{(0)} = \bar{x}$ and $\bar{x}^{(i+1)} = M_k(\bar{x}^{(i)})$. Since M_k consists of three stages T_1^k , T_2^k and T_3^k , we extend the notation to describe the output of each stage: $\bar{x}^{(i,0)} = \bar{x}^{(i)}$ and $\bar{x}^{(i,j)} = T_j^k(\bar{x}^{(i,j-1)})$, for $j = 1, 2, 3$. For other values of j we assume that $\bar{x}^{(i,j)} = \bar{x}^{(i+j \div 3, j \bmod 3)}$. We will use this superscript notation for other equivalent representations of sequence \bar{x} .

Now let us fix some technical notations and definitions. A 0-1 sequence can be represented as a word over $\Sigma = \{0, 1\}$. A non-decreasing (also called *sorted*) 0-1 sequence has a form of 0^*1^* and can be equivalently represented by the number of ones (or zeros) in it. For any $x \in \Sigma^*$ let $\text{ones}(x)$ denote the number of 1 in x . If $x \in \Sigma^n$ then x_i , $1 \leq i \leq n$, denotes the i -th letter of x and x_A , $A = \{i_1, \dots, i_m\}$, $1 \leq i_1 < \dots < i_m \leq n$ denotes the word $x_{i_1} \dots x_{i_m}$. We say that a 0-1 sequence $\bar{x} = (x_1, \dots, x_{N_k})$ is *2-sorted* if both $(x_1, x_3, \dots, x_{N_k-1})$ and $(x_2, x_4, \dots, x_{N_k})$ are sorted.

3.1 Reduction to Analysis of Columns

For any $k \geq 3$ let $n_k = 2^{k-1} - 1$, $b_k = 2(k-2)$ (thus $N_k = n_k \cdot b_k$). The set of registers $\text{Reg}_k = \{1, \dots, N_k\}$ can be analyzed as an $n_k \times b_k$ matrix with $C_j^k = \{j + ib_k : 0 \leq i < n_k\}$, $j = 1, \dots, b_k$, as columns. A content of all registers in the matrix, that is $x \in \Sigma^{N_k}$, can be equivalently represented by the sequence of contents of registers in C_1, C_2, \dots, C_{b_k} , that is $(x_{C_1}, \dots, x_{C_{b_k}})$. Since b_k is an even number, the following fact is obviously true.

Fact 3 *If $x \in \Sigma^{N_k}$ is 2-sorted then each x_{C_j} , $j = 1, \dots, b_k$, is sorted.* \square

That is, the columns are sorted at the beginning of a computation of $2k-5$ passes of M_k . The first lemma we would like to prove is that columns remain sorted after each stage of the computation. We start with a following technical fact:

Fact 4 *Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be subsets of $\{1, \dots, N_k\}$ such that $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$. Let $h \geq 0$ and $S_{A,B,h} = \{[a_i : b_{i+h}] : 1 \leq i \leq n-h\}$. Then for any $x \in \Sigma^{N_k}$ such that x_A and x_B are sorted, the output $y = S_{A,B,h}(x)$ has the following properties:*

- (i) y_A and y_B are sorted.
- (ii) Let $m_1 = \text{ones}(x_A)$ and $m_2 = \text{ones}(x_B)$. Then $\text{ones}(y_A) = \min(m_1, m_2 + h)$ and $\text{ones}(y_B) = \max(m_1 - h, m_2)$.

Proof. To prove (i) we show only that $y_{a_i} \leq y_{a_{i+1}}$ for $i = 1, \dots, n-1$. If $1 \leq i < n-h$ then $y_{a_i} = \min(x_{a_i}, x_{b_{i+h}}) \leq \min(x_{a_{i+1}}, x_{b_{i+h+1}}) = y_{a_{i+1}}$ since \min is a non-decreasing function and both x_A and x_B are sorted. If $i = n-h$ then $y_{a_i} = \min(x_{a_i}, x_{b_{i+h}}) \leq x_{a_{i+1}} = y_{a_{i+1}}$. For $i > n-h$ we have $y_{a_i} = x_{a_i} \leq x_{a_{i+1}} = y_{a_{i+1}}$.

To prove (ii) let $m'_1 = \min(m_1, m_2 + h)$ and $m'_2 = \max(m_1 - h, m_2)$. We consider two cases. If $m_1 \leq m_2 + h$ then $m_1 - h \leq m_2$ and we get $m'_1 = m_1$ and $m'_2 = m_2$. In this case no comparator from $S_{A,B,h}$ exchanges 0 with 1. To see this assume a.c. that a comparator $[a_i : b_{i+h}]$ exchanges $x_{a_i} = 1$ with $x_{b_{i+h}} = 0$. Then $i > n - m_1$ and $i + h \leq n - m_2$ hold because of the definitions of m_1 and m_2 . It follows that $n - m_1 < n - m_2 - h$, thus $m_1 - h > m_2$ — a contradiction. If $m_1 > m_2 + h$ then $m'_1 = m_2 + h$ and $m'_2 = m_1 - h$. In this case let us observe that a comparator $[a_i : b_{i+h}]$ exchanges $x_{a_i} = 1$ with $x_{b_{i+h}} = 0$ if and only if $m_2 + h \leq n - i < m_1$. Therefore $\text{ones}(y_A) = m_1 - (m_1 - m_2 - h) = m_2 + h$ and $\text{ones}(y_B) = m_2 + (m_1 - m_2 - h) = m_1 - h$. \square

According to the definition of M_k , it consists of three stages T_1^k, T_2^k, T_3^k , where $T_i^k = \cup \{S_{k,i+3j} : 0 \leq j \leq \lfloor \frac{2k-i-3}{3} \rfloor\}$ (sets S_j are defined in Def. 1). Using the notation from Fact 4, the following fact is an easy consequence of Definition 1.

Fact 5 Let $L_i = C_i$ and $R_i = C_{b_k-i+1}$ denote the corresponding left and the right columns of registers, and $h_i = 2^{k-i-1} - 1$, $i = 1, \dots, \frac{b_k}{2}$. Then

- (i) $\text{regs}(S_{k,1}) \subseteq L_1 \cup R_1$ and $S_{k,1} = S_{R_1 - \{N_k\}, L_1 - \{1\}, 0}$
- (ii) $\text{regs}(S_{k,2j}) \subseteq L_j \cup R_j$ and $S_{k,2j} = S_{L_j, R_j, h_j}$, for any $j = 1, \dots, \frac{b_k}{2}$
- (iii) $\text{regs}(S_{k,2j+1}) \subseteq L_j \cup L_{j+1} \cup R_{j+1} \cup R_j$ and $S_{2j+1} = S_{L_j, L_{j+1}, 0} \cup S_{R_{j+1}, R_j, 0}$, for any $j = 1, \dots, \frac{b_k}{2} - 1$
- (iv) $\text{regs}(S_{k,2k-3}) \subseteq L_{k-2} \cup R_{k-2}$ and $S_{k,2k-3} = S_{L_{k-2}, R_{k-2}, 0}$
- (v) if $(L_j \cup R_j) \cap \text{regs}(S_{k,i}) \neq \emptyset$ then $2j - 1 \leq i \leq 2j + 1$, for any $j = 1, \dots, \frac{b_k}{2} - 1$

\square

Lemma 1. If the initial content of registers is a 2-sorted 0-1 sequence x then after each stage of multi-pass computation of $M_k = T_1^k, T_2^k, T_3^k$ the content of each column C_j , $j = 1, \dots, b_k$, is sorted, that is, each $(x^{(p,i)})_{C_j}$ is of the form 0^*1^* , $p = 0, \dots, i = 1, 2, 3$.

Proof. By induction it suffices to prove that for each sequence $y \in \Sigma^{N_k}$ with sorted columns C_j , $j = 1, \dots, b_k$, the outputs $z_i = T_i^k(y)$, $i = 1, 2, 3$ have also the columns sorted. Since each T_i^k , as a mapping, is a composition of mapping $S_{i+3j}, 0 \leq j \leq \lfloor \frac{2k-i-3}{3} \rfloor$, each of which, due to Facts 4 and 5, transforms sorted columns into sorted columns, the lemma follows. \square

From now on, instead of looking at 0-1 sequences with sorted columns, we will analyze the computations of M_k on sequences of integers $\vec{c} = (c_1, \dots, c_{b_k})$, where c_t , $t = 1, \dots, b_k$, denote the number of ones in a sorted column C_t . Transformations of 0-1 sequences defined by sets S_j , $j = 1, \dots, 2k - 3$ will be represented by the following mappings:

Definition 3. Let $k \geq 3$, $h_i = 2^{k-i-1} - 1$ for $i = 1, \dots, k - 2$ and $b_k = 2(k - 2)$. The functions dec_i^k , mov_i^k and cyc^k over sequences of b_k reals are defined as follows. Let $\vec{c} = (c_1, \dots, c_{b_k})$ and $t \in \{1, \dots, b_k\}$.

$$(\text{dec}_i^k(\vec{c}))_t = \begin{cases} \min(c_i, c_{b_k-i+1} + h_i) & \text{if } t = i \\ \max(c_i - h_i, c_{b_k-i+1}) & \text{if } t = b_k - i + 1 \\ c_t & \text{otherwise} \end{cases} \quad (10)$$

$$(mov_i^k(\bar{c}))_t = \begin{cases} \min(c_t, c_{t+1}) & \text{if } t = i \text{ or } t = b_k - i \\ \max(c_{t-1}, c_t) & \text{if } t = i + 1 \text{ or } t = b_k - i + 1 \\ c_t & \text{otherwise} \end{cases} \quad (11)$$

$$(cyc^k(\bar{c}))_t = \begin{cases} \max(c_1, c_{b_k} - 1) & \text{if } t = 1 \\ \min(c_1 + 1, c_{b_k}) & \text{if } t = b_k \\ c_t & \text{otherwise} \end{cases} \quad (12)$$

Fact 6 Let $x \in \Sigma^{N_k}$ be a 0-1 sequence with sorted columns C_1, \dots, C_{b_k} , let $c_i = \text{ones}(x_{C_i})$ and $\bar{c} = (c_1, \dots, c_{b_k})$. Let $y_j = S_{k,j}(x)$, $d_{j,i} = \text{ones}((y_j)_{C_i})$ and $\bar{d}_j = (d_{j,1}, \dots, d_{j,b_k})$, where $i = 1, \dots, b_k$ and $j = 1, \dots, 2k - 3$. Then

- (i) $\bar{d}_1 = cyc^k(\bar{c})$
- (ii) $\bar{d}_{2j} = dec_j^k(\bar{c})$, for any $j = 1, \dots, \frac{b_k}{2}$
- (iii) $\bar{d}_{2j+1} = mov_j^k(\bar{c})$, for any $j = 1, \dots, \frac{b_k}{2}$

Proof. Generally, the fact follows from Fact 5 and the part (ii) of Fact 4. We prove only its parts (i) and (ii). Part (iii) can be proved in the similar way.

(i) Observe that $y_1 = S_{k,1}(x) = S_{R_1 - \{N_k\}, L_1 - \{1\}, 0}(x)$ due to Fact 5 (ii). It follows that only the content of columns $L_1 = C_1$ and $R_1 = C_{b_k}$ can change, but they remain sorted (according to Lemma 1). Using Fact 4 (ii) we have: $m_1 = \text{ones}(x_{R_1 - \{N_k\}}) = c_{b_k} - x_{N_k}$, $m_2 = \text{ones}(x_{L_1 - \{1\}}) = c_1 - x_1$ and

$$d_{1,1} = \max(m_1, m_2) + x_1 = \max(c_{b_k} - x_{N_k} + x_1, c_1),$$

$$d_{1,b_k} = \min(m_1, m_2) + x_{N_k} = \min(c_{b_k}, c_1 + x_{N_k} - x_1).$$

Now let us consider the following three cases of values x_1 and x_{N_k} :

Case $x_1 = 0$ and $x_{N_k} = 1$. Then $d_{1,1} = \max(c_{b_k} - 1, c_1) = cyc^k(\bar{c})_1$ and $d_{1,b_k} = \min(c_{b_k}, c_1 + 1) = cyc^k(\bar{c})_1$.

Case $x_1 = 1$. Then $c_1 = n_k$, $c_{b_k} \leq n_k$ and $c_{b_k} - x_{N_k} \leq n_k - 1$. In this case: $d_{1,1} = \max(n_k, c_{b_k} - x_{N_k} + 1) = n_k = \max(c_1, c_{b_k} - 1)$ and $d_{1,b_k} = \min(n_k - 1 + x_{N_k}, c_{b_k}) = c_{b_k} = \min(c_1 + 1, c_{b_k})$.

Case $x_{N_k} = 0$. Then $c_{b_k} = 0$ and $c_1 - x_1 \geq 0$. In this case: $d_{1,1} = \max(c_1, x_1) = c_1 = \max(c_1, c_{b_k} - 1)$ and $d_{1,b_k} = \min(c_1 - x_1, c_{b_k}) = c_{b_k} = \min(c_1 + 1, c_{b_k})$.

(ii) We fix any $j \in \{1, \dots, \frac{b_k}{2}\}$ and observe that $y_{2j} = S_{2j}(x) = S_{L_j, R_j, h_j}(x)$ due to Fact 5 (ii). It follows that only the content of columns $L_j = C_j$ and $R_j = C_{b_k - j + 1}$ can change, but they remain sorted (according to Lemma 1). Using Fact 4 (ii) we have:

$$d_{2j,j} = \text{ones}((y_{2j})_{L_j}) = \min(c_j, c_{b_k - j + 1} + h_j) = (dec_j^k(\bar{c}))_j,$$

$$d_{2j,b_k - j + 1} = \text{ones}((y_{2j})_{R_j}) = \max(c_j - h_j, c_{b_k - j + 1}) = (dec_j^k(\bar{c}))_{b_k - j + 1}.$$

Definition 4. Let $k \geq 3$. Let Q_1^k , Q_2^k and Q_3^k denote the following sets of functions.

$$Q_1^k = \{cyc^k\} \cup \{dec_{3i-1}^k\}_{i=1}^{\lfloor \frac{k-1}{3} \rfloor} \cup \{mov_{3i}^k\}_{i=1}^{\lfloor \frac{k-2}{3} \rfloor} \quad (13)$$

$$Q_2^k = \{dec_{3i-2}^k\}_{i=1}^{\lfloor \frac{k}{3} \rfloor} \cup \{mov_{3i-1}^k\}_{i=1}^{\lfloor \frac{k-1}{3} \rfloor} \quad (14)$$

$$Q_3^k = \{dec_{3i}^k\}_{i=1}^{\lfloor \frac{k-2}{3} \rfloor} \cup \{mov_{3i-2}^k\}_{i=1}^{\lfloor \frac{k}{3} \rfloor} \quad (15)$$

Let us observe that each function in Q_i^k , $i = 1, 2, 3$, can only modify a few positions in a given sequence of numbers. Moreover, different functions in Q_i^k can only modify disjoint sets of positions. For a function $f : R^m \mapsto R^m$ let us define

$$\text{args}(f) = \{i \in \{1, \dots, m\} : \exists \bar{c} \in R^m (f(\bar{c}))_i \neq (\bar{c})_i\}$$

The following facts formalize our observations.

Fact 7 $\text{args}(\text{cyc}^k) = \{1, b_k\}$, $\text{args}(\text{dec}_i^k) = \{i, b_k - i + 1\}$, $\text{args}(\text{mov}_i^k) = \{i, i + 1, b_k - i, b_k - i + 1\}$, where $i = 1, \dots, k - 2$. □

Fact 8 For each pair of functions $f, g \in Q_i^k$, $f \neq g$, $i = 1, 2, 3$, we have

- (i) $\text{args}(f) \cap \text{args}(g) = \emptyset$;
- (ii) for any $\bar{c} = (c_1, \dots, c_{b_k})$ and $j \in \{1, \dots, b_k\}$

$$(f(g(\bar{c})))_j = \begin{cases} (f(\bar{c}))_j & \text{if } j \in \text{args}(f) \\ (g(\bar{c}))_j & \text{if } j \in \text{args}(g) \\ c_j & \text{otherwise} \end{cases} \quad (16)$$

□

Corollary 1. Each set Q_i^k , $i = 1, 2, 3$, uniquely determines a mapping, in which functions from Q_i^k can be apply in any order. Moreover, if $f \in Q_i^k$, $\bar{c} \in R^{b_k}$ and $j \in \text{args}(f)$ then $(Q_i^k(\bar{c}))_j = (f(\bar{c}))_j$.

We would like to prove that the result of applying Q_i^k , $i = 1, 2, 3$, to a sequence $\bar{c} = (c_1, \dots, c_{b_k})$ of numbers of ones in columns C_1, \dots, C_{b_k} is equivalent to applying the set of comparators T_i^k to the content of registers, if each column is sorted.

Lemma 2. Let $x \in \Sigma^{N_k}$ be a 0-1 sequence with sorted columns C_1, \dots, C_{b_k} , let $c_i = \text{ones}(x_{C_i})$ and $\bar{c} = (c_1, \dots, c_{b_k})$. Let $y_j = T_j^k(x)$, $d_{j,i} = \text{ones}((y_j)_{C_i})$ and $\bar{d}_j = (d_{j,1}, \dots, d_{j,b_k})$, where $i = 1, \dots, b_k$ and $j = 1, 2, 3$. Then $Q_j^k(\bar{c}) = \bar{d}_j$.

Proof. Recall that $T_j^k = \bigcup \{S_{k,j+3i} : 0 \leq i \leq \frac{2k-j-3}{3}\}$. For a set of comparators S let us define

$$\text{cols}(S) = \{i \in \{1, \dots, b_k\} : \text{regs}(S) \cap C_i \neq \emptyset\} .$$

From Fact 5(i-iv) it follows that $\text{cols}(S_{k,1}) = \{1, b_k\}$ and for $i = 1, \dots, k - 2$ $\text{cols}(S_{k,2i}) = \{i, b_k - i + 1\}$ and $\text{cols}(S_{k,2i+1}) = \{i, i + 1, b_k - i, b_k - i + 1\}$. From Fact 5(v) we get that $\text{cols}(S_{k,j+3i}) \cap \text{cols}(S_{k,j+3i'}) = \emptyset$ if $i \neq i'$. Thus we can observe a 1-1 correspondence between a function f in Q_j^k and a set of comparators $S_{k,j+3i} \subseteq T_j^k$ such that $\text{args}(f) = \text{cols}(S_{k,j+3i})$. Then for each $t \in \text{args}(f)$ $(Q_j^k(\bar{c}))_t = (f(\bar{c}))_t = (\bar{d}_j)_t$, as the consequence of Corollary 1 and Fact 6. □

Definition 5. We say that a sequence of numbers $\bar{c} = (c_1, \dots, c_{2m})$ is flat if $c_1 \leq c_2 \leq \dots, c_{2m} \leq c_1 + 1$. We say that a sequence \bar{c} is 2-flat if subsequences $(c_1, c_3, \dots, c_{2m-1})$ and $(c_2, c_4, \dots, c_{2m})$ are flat. We say that \bar{c} is balanced if $c_i + c_{2m-i+1} = c_1 + c_{2m}$, for $i = 2, \dots, m$. For a balanced sequence \bar{c} define $\text{height}(\bar{c})$ as $c_1 + c_{2m}$.

Proposition 3. Let $k \geq 3$, $x \in \Sigma^{N_k}$, $\bar{c} = (c_1, \dots, c_{b_k})$, where $c_i = \text{ones}(x_{C_i})$ (C_i is as usual a column in the matrix of registers), $i = 1, \dots, b_k$. Then

1. x is sorted if and only if columns of x are sorted and \bar{c} is flat;
2. x is 2-sorted if and only if columns of x are sorted and \bar{c} is 2-flat;

□

Now we are ready to reduce the proof of Theorem 2 to the proof of following lemma.

Lemma 3. Let $k \geq 3$. If for each 2-flat sequence $\bar{c} = (c_1, \dots, c_{b_k})$ of integers from $[0, 2^{k-1} - 1]$ the result of application $(Q_3^k \circ Q_2^k \circ Q_1^k)^{2k-5}$ to (\bar{c}) is a flat sequence, then M_k is a $2k - 5$ -pass merger of two sorted sequences given in odd and even registers, respectively.

Proof. Assume that for each 2-flat sequence $\bar{c} = (c_1, \dots, c_{b_k})$ the result of application $(Q_3^k \circ Q_2^k \circ Q_1^k)^{2k-5}$ to (\bar{c}) is a flat sequence. Let $\bar{x} \in \Sigma^{N_k}$ be a 2-sorted sequence and $\bar{c} = (c_1, \dots, c_{b_k})$, where $c_i = \text{ones}(\bar{x}_{C_i})$ (C_i is as usual a column in the matrix of registers), $i = 1, \dots, b_k$. Then \bar{c} is 2-flat due to Proposition 3 and each $c_i \in [0, 2^{k-1} - 1]$, because the height of columns is $2^{k-1} - 1$. Recall that $\bar{x}^{(j)} = (M_k)^j(\bar{x})$ and let $c_{j,i} = \text{ones}(\bar{x}_{C_i}^{(j)})$. Using Lemma 2 and easy induction we get that the equality $(Q_3^k \circ Q_2^k \circ Q_1^k)^j(\bar{c}) = (c_{j,1}, \dots, c_{j,b_k})$ is true for $j = 1, \dots, 2k - 5$. Since $(Q_3^k \circ Q_2^k \circ Q_1^k)^{2k-5}(\bar{c})$ is a flat sequence, the sequence $\bar{x}^{(2k-5)}$ is sorted. □

3.2 Analysis of Balanced Columns

Due to Lemma 3 we can only analyze the results of periodic application of the functions Q_1^k , Q_2^k and Q_3^k to a sequence of integers representing the numbers of ones in each register column. We know also that an initial sequence is 2-flat. To simplify our analysis further, we start it with initial values restricted to be balanced 2-flat sequences. Then we observe that the functions are monotone and any 2-flat sequence can be bounded from below and above by balanced 2-flat sequences whose heights differ only by one.

Lemma 4. Let $k \geq 3$ and $\bar{c} = (c_1, \dots, c_{b_k})$ be a balanced sequence of numbers. Let $s = \text{height}(\bar{c})$ and let f be a function from $Q_1^k \cup Q_2^k \cup Q_3^k$. Then $f(\bar{c})$ is also balanced and $\text{height}(f(\bar{c})) = s$.

Proof. Let \bar{c} and s be as in Lemma and let $f(\bar{c}) = (d_1, \dots, d_{b_k})$. The function $f \in Q_1^k \cup Q_2^k \cup Q_3^k$ can be either cyc^k or one of $\text{mov}_j^k, \text{dec}_j^k$, $j = 1, \dots, k - 2$, according to Definition 4. Each of the functions can only modify one or two pairs of positions of the form $(i, b_k - i + 1)$ in \bar{c} (see Definition 3). The other pairs are left untouched, so the sum of their values cannot change. In case of cyc^k the modified pair is $(1, b_k)$ and $d_1 + d_{b_k} = \max(c_1, c_{b_k} - 1) + \min(c_1 + 1, c_{b_k}) = s$. In case of dec_j^k the pair is $(j, b_k - j + 1)$ and $d_j + d_{b_k - j + 1} = \min(c_j, c_{b_k - j + 1} + h_j) + \max(c_j - h_j, c_{b_k - j + 1}) = \min(c_j - h_j, c_{b_k - j + 1}) + h_j + \max(c_j - h_j, c_{b_k - j + 1}) = s$. Finally, if $f = \text{mov}_j^k$ then we have two pairs $(j, b_k - j + 1)$ and $(j + 1, b_k - j)$. Then $d_j + d_{b_k - j + 1} = \min(c_j, c_{j+1}) + \max(c_{b_k - j}, c_{b_k - j + 1}) = \min(c_j, c_{j+1}) + \max(s - c_{j+1}, s - c_j) = s$ and in case of the second pair $d_{j+1} + d_{b_k - j} = \max(c_j, c_{j+1}) + \min(c_{b_k - j}, c_{b_k - j + 1}) = \max(c_j, c_{j+1}) + \min(s - c_{j+1}, s - c_j) = s$. □

It follows from Lemma 4 that if we start the periodical application of the functions Q_1^k , Q_2^k and Q_3^k to a balanced 2-flat initial sequence then it remains balanced after each function application and its height will not changed. Therefore, we can only trace the values in the first half of generated sequences. If needed, a value in the second half can be computed from the height and the corresponding value in the first half. To get a better view on the structure of generated sequences, we subtract half of the height from each element of the initial sequence and proceed with such modified sequences to the end. At the end the subtracted value is added to each element of the final sequence. The following fact justifies the described above procedure.

Fact 9 *Let f be a function from $Q_1^k \cup Q_2^k \cup Q_3^k$. Then f is monotone and for each $t \in R$ and (c_1, \dots, c_{b_k}) the following equation is true*

$$f(c_1 - t, \dots, c_{b_k} - t) = f(c_1, \dots, c_{b_k}) - (t, \dots, t) .$$

Proof. The fact follows from the similar properties of min and max functions: they are monotone and the equations: $\min(x - t, y - t) = \min(x, y) - t$ and $\max(x - t, y - t) = \max(x, y) - t$ are obviously true. Each f in $Q_1^k \cup Q_2^k \cup Q_3^k$ is defined with the help of these simple functions, thus f inherits the properties. \square

Corollary 2. *Let $f = f_l \circ f_{l-1} \circ \dots \circ f_1$, where $f_i \in \{Q_1^k, Q_2^k, Q_3^k\}$, $1 \leq i \leq l$. Then f is monotone and for any $t \in R$ and $(c_1, \dots, c_{b_k}) \in R^{b_k}$*

$$f(c_1 - t, \dots, c_{b_k} - t) = f(c_1, \dots, c_{b_k}) - (t, \dots, t) .$$

\square

Definition 6. *Let $\bar{c} = (c_1, \dots, c_{b_k}) \in R^{b_k}$ be a balanced sequence and $s = \text{height}(\bar{c})$. We call $(c_1 - \frac{s}{2}, c_2 - \frac{s}{2}, \dots, c_{b_k} - \frac{s}{2}) \in R^{b_k/2}$ the reduced sequence of \bar{c} and denote it by $\text{reduce}(\bar{c})$. For a sequence $\bar{d} = (d_1, \dots, d_{k-2}) \in R^{k-2}$ we define s -extended sequence $\text{ext}(\bar{d}, s)$ as*

$$(d_1 + \frac{s}{2}, d_2 + \frac{s}{2}, \dots, d_{k-2} + \frac{s}{2}, \frac{s}{2} - d_{k-2}, \frac{s}{2} - d_{k-3}, \dots, \frac{s}{2} - d_1) .$$

For any $t \in R$ and a function $f : R^{b_k} \mapsto R^{b_k}$ that maps each balanced sequence to a balanced one and preserves its height let $\text{reduce}(f, t)$ denote a function on R^{k-2} such that $(\text{reduce}(f, t))(\bar{d}) = \text{reduce}(f(\text{ext}(\bar{d}, t)))$ for any $\bar{d} \in R^{k-2}$.

Observe that for a balanced sequence \bar{c} with height s the sequence $\text{ext}(\text{reduce}(\bar{c}), s)$ is equal to \bar{c} . Moreover, for any $t \in R$ and a sequence $\bar{d} \in R_{k-2}$ the sequence $\text{ext}(\bar{d}, t)$ is balanced and its height is t , thus $\text{reduce}(\text{ext}(\bar{d}, t)) = \bar{d}$. Note also that functions Q_1^k , Q_2^k and Q_3^k preserve the property of being balanced and the sequence height (see Lemma 4), so we can analyze a periodical application of their reduced forms to a reduced balanced 2-flat input.

Fact 10 *Let $f = f_l \circ f_{l-1} \circ \dots \circ f_1$, where $f_i \in \{Q_1^k, Q_2^k, Q_3^k\}$, $1 \leq i \leq l$. Let $\bar{c} \in R^{b_k}$ be balanced and $s = \text{height}(\bar{c})$. Let $\hat{f}_i = \text{reduce}(f_i, s)$, $1 \leq i \leq l$. Then $f(\bar{c}) = \text{ext}((\hat{f}_l \circ \hat{f}_{l-1} \circ \dots \circ \hat{f}_1)(\text{reduce}(\bar{c})), s)$.* \square

Definition 7. Let $MinMax(x, y) = (\min(x, y), \max(x, y))$, $Min(x) = \min(x, -x)$, $Cyc(x) = \max(x, -x - 1)$ and $Dec_i(x) = \min(x, -x + H_i)$, where $H_i = 2^i - 1, i = 1, \dots$. Moreover, let us define the following sequences of functions:

$$\hat{Q}_1^k = (Cyc) \oplus \bigoplus_{i=1}^{\lfloor \frac{k-3}{3} \rfloor} (Dec_{k-3i}, MinMax) \oplus (F_1^k) \quad (17)$$

$$\hat{Q}_2^k = \bigoplus_{i=1}^{\lfloor \frac{k-2}{3} \rfloor} (Dec_{k-3i+1}, MinMax) \oplus (F_2^k) \quad (18)$$

$$\hat{Q}_3^k = \bigoplus_{i=1}^{\lfloor \frac{k-2}{3} \rfloor} (MinMax, Dec_{k-3i-1}) \oplus (F_3^k), \quad (19)$$

where \oplus denote concatenation of sequences and for $i = 1, 2$

$$F_i^k = \begin{cases} () & \text{if } k \equiv 2i + 1 \pmod{3} \\ (Dec_1) & \text{if } k \equiv 2i + 2 \pmod{3} \\ (Dec_2, Min) & \text{if } k \equiv 2i \pmod{3} \end{cases} \quad F_3^k = \begin{cases} () & \text{if } k \equiv 2 \pmod{3} \\ (Min) & \text{if } k \equiv 0 \pmod{3} \\ (MinMax) & \text{if } k \equiv 1 \pmod{3} \end{cases}$$

Lemma 5. Let $k \geq 3$ and $t \in R$. Then $reduce(Q_i^k, t) = \otimes \hat{Q}_i^k$, where $i = 1, 2, 3$ and \otimes denotes the Cartesian product of a sequence of functions.

Proof. Let $k \geq 3, i \in \{1, 2, 3\}$ and $t \in R$. Let $\bar{d} \in R^{k-2}$. By Def. 6, $(reduce(Q_i^k, t))(\bar{d}) = reduce(Q_i^k(ext(\bar{d}, t)))$. Let $\bar{e} = ext(\bar{d}, t) = (d_1 + \frac{t}{2}, \dots, d_{k-2} + \frac{t}{2}, -d_{k-2} + \frac{t}{2}, \dots, -d_1 + \frac{t}{2})$. The sequence \bar{e} is balanced and $height(\bar{e}) = t$. To get the lemma we would like to prove that for $j = 1, \dots, k-2$ the equalities $(Q_i^k(\bar{e}))_j - \frac{t}{2} = ((\otimes \hat{Q}_i^k)(\bar{d}))_j$ hold. The proof is by case analysis of values of i and j . In the following equations we use Definitions 3, 4, 6 and 7.

1. (Case: $i = 1$ and $j = 1$). Then $(Q_1^k(\bar{e}))_1 = (cyc^k(\bar{e}))_1 = \max(d_1 + \frac{t}{2}, -d_1 + \frac{t}{2} - 1) = \max(d_1, -d_1 - 1) + \frac{t}{2} = Cyc(d_1) + \frac{t}{2} = ((\otimes \hat{Q}_1^k)(\bar{d}))_1 + \frac{t}{2}$.
2. (Case: $i + j > 2$ and $i + j \equiv 0 \pmod{3}$). Let l be such that $j = 3l - i$. Then $(Q_i^k(\bar{e}))_j = (dec_{3l-i}^k(\bar{e}))_{3l-i} = \min(d_{3l-i} + \frac{t}{2}, -d_{3l-i} + \frac{t}{2} + 2^{k-(3l-i)-1} - 1) = \min(d_{3l-i}, -d_{3l-i} + H_{k-(3l-i)-1}) + \frac{t}{2} = Dec_{k-3l+i-1}(d_{3l-i}) + \frac{t}{2} = ((\otimes \hat{Q}_i^k)(\bar{d}))_j + \frac{t}{2}$.
3. (Case: $i + j > 2, j < k-2$ and $i + j \equiv 1 \pmod{3}$). Let l be such that $j = 3l - i + 1$. Then $(Q_i^k(\bar{e}))_j = (mov_{3l-i+1}^k(\bar{e}))_{3l-i+1} = \min(d_{3l-i+1} + \frac{t}{2}, d_{3l-i+2} + \frac{t}{2}) = \min(d_{3l-i+1}, d_{3l-i+2}) + \frac{t}{2}$. Starting from the other side we get $((\otimes \hat{Q}_i^k)(\bar{d}))_{3l-i+1} = (MinMax(d_{3l-i+1}, d_{3l-i+2}))_1 = \min(d_{3l-i+1}, d_{3l-i+2})$ and we are done.
4. (Case: $i + j > 2, j = k-2$ and $i + j \equiv 1 \pmod{3}$). Let l be as in previous case. Then $(Q_i^k(\bar{e}))_{k-2} = (mov_{k-2}^k(\bar{e}))_{k-2} = \min(d_{k-2} + \frac{t}{2}, -d_{k-2} + \frac{t}{2}) = \min(d_{k-2}, -d_{k-2}) + \frac{t}{2} = Min(d_{k-2}) + \frac{t}{2} = ((\otimes \hat{Q}_i^k)(\bar{d}))_{k-2} + \frac{t}{2}$.
5. (Case: $i + j > 2$ and $i + j \equiv 2 \pmod{3}$). Let l be such that $j = 3l - i + 2$. Then $(Q_i^k(\bar{e}))_{3l-i+2} = (mov_{3l-i+1}^k(\bar{e}))_{3l-i+2} = \max(d_{3l-i+1} + \frac{t}{2}, d_{3l-i+2} + \frac{t}{2}) = \max(d_{3l-i+1}, d_{3l-i+2}) + \frac{t}{2}$. Starting from the other side we get $((\otimes \hat{Q}_i^k)(\bar{d}))_{3l-i+2} = (MinMax(d_{3l-i+1}, d_{3l-i+2}))_2 = \max(d_{3l-i+1}, d_{3l-i+2})$ and we are finally done. \square

Instead of tracing individual values in reduced sequences after each application of a function from $\{\otimes \hat{Q}_1^k, \otimes \hat{Q}_2^k, \otimes \hat{Q}_3^k\}$ we will trace intervals in which the values should be and observe how the lengths of intervals are decreasing during the computation. So let us now define the intervals and show a fact about computations on them.

Definition 8. Let $k \geq 3$, $H_i = 2^i - 1$ for $1 \leq i \leq k-1$. Let $I(0)$ denote the interval $[-\frac{1}{2}, 0]$ and, in similar way, let $I(i) = [-\frac{1}{2}, \frac{H_i}{2}]$, $1 \leq i \leq k-1$, $I(-k) = [-\frac{H_{k-1}}{2}, 0]$ and $I(\pm k) = [-\frac{H_{k-1}}{2}, \frac{H_{k-1}}{2}]$. Moreover, we will write $I(w_1, w_2, \dots, w_l)$ for the Cartesian product $I(w_1) \times I(w_2) \times \dots \times I(w_l)$, where each $w_i \in \{0, 1, 2, \dots, k-1, -k, \pm k\}$.

Fact 11 The following inclusions are true:

1. $Dec_i(I(i+1)) \subseteq I(i)$ and $Dec_i(I(w)) \subseteq I(w)$, for $1 \leq i \leq k-2$ and $w \in \{0, -k, \pm k\}$;
2. $Cyc(I(-k)) \subseteq I(k-1)$ and $Cyc(w) \subseteq Cyc(w)$, for $w \in \{0, k-1\}$;
3. $Min(I(\pm k)) \subseteq I(-k)$ and $Min(I(1)) \subseteq I(0)$;
4. $MinMax(I(\pm k, -k)) \subseteq (I(-k, \pm k))$;
5. $MinMax(I(i, w)) \subseteq (I(w, i))$, for $1 \leq i \leq k-1$ and $w \in \{0, -k\}$.

Proof. The proof of each inclusion is a straightforward consequence of the definitions of a given function and intervals. Therefore we check only inclusions given in the first item. Let $x \in I(i+1) = [-\frac{1}{2}, \frac{H_{i+1}}{2}]$. If $x \in I(i) = [-\frac{1}{2}, \frac{H_i}{2}]$, then $Dec_i(x) = \min(x, -x + H_i) = x$ since $2x \leq H_i$. Otherwise x must be in $(\frac{H_i}{2}, \frac{H_{i+1}}{2}]$, but then $x > -x + H_i$ and $Dec_i(x) = -x + H_i \in [-\frac{1}{2}, \frac{H_i}{2}]$ since $H_{i+1} = 2H_i + 1$.

To proof the second inclusion for Dec_i let us observe that if $x \leq 0$ then $Dec_i(x) = x$. It follows that $Dec_i(I(0)) \subseteq I(0)$ and $Dec_i(I(-k)) \subseteq I(-k)$. In case of $x \in I(\pm k)$ we only have to check the positive values of x , such that $x \geq -x + H_i$. But then $Dec_i(x) = -x + H_i > -x$ and both $x, -x \in I(\pm k)$. \square

Now we are ready to define sequences of intervals that are used to describe states of computation after each periodic application of functions \hat{Q}_1^k , \hat{Q}_2^k and \hat{Q}_3^k to a reduced sequence of numbers of ones in columns.

Definition 9. Let $k \geq 3$. By Z^k we denote the sequence $(0, 0, 0)^{\lceil \frac{k-2}{3} \rceil}$ and, in the similar way, $U_1^k = (\pm k, \pm k, -k)^{\lceil \frac{k-2}{3} \rceil}$, $U_2^k = (\pm k, -k, \pm k)^{\lceil \frac{k-2}{3} \rceil}$ and $U_0^k = (-k, \pm k, \pm k)^{\lceil \frac{k-2}{3} \rceil}$.

Next, let $V_1^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (k-3i+2, k-3i, -k)$, $V_2^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (k-3i+1, -k, k-3i)$ and let $V_0^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (-k, k-3i+1, k-3i-1)$.

Finally, let $W_1^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (k-3i+2, k-3i, 0)$, $W_2^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (k-3i+1, 0, k-3i)$ and let $W_0^k = \bigoplus_{i=1}^{\lceil \frac{k-2}{3} \rceil} (0, k-3i+1, k-3i-1)$.

Note that all sequences defined above are of length $3\lceil \frac{k-2}{3} \rceil \geq k-2$ and their elements are interval descriptors as defined in Definition 8.

Definition 10. Let $k \geq 3$. Let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be any sequences, where $n \geq k-2$. For $0 \leq i \leq k-2$ let $join_k(i, \bar{a}, \bar{b})$ denote $(a_1, \dots, a_i, b_{i+1}, \dots, b_{k-2})$.

Definition 11. Let $k \geq 3$. Let X_i^k denote a state sequence after i stages and be defined as:

$$X_i^k = \begin{cases} \text{join}_k(\lceil \frac{i+1}{2} \rceil, V_{i \bmod 3}^k, U_{i \bmod 3}^k) & \text{for } i = 1, \dots, 2k-5 \\ \text{join}_k(3k-6-i, V_{i \bmod 3}^k, W_{i \bmod 3}^k) & \text{for } i = 2k-4, \dots, 3k-7 \\ \text{join}_k(\lceil \frac{i+1-(3k-6)}{2} \rceil, Z^k, W_{i \bmod 3}^k) & \text{for } i = 3k-6, \dots, 5k-12 \end{cases}$$

For example, to create X_1^k we take the first element of V_1^k and the rest of elements from U_1^k obtaining the sequence $(k-1, \pm k, -k, \pm k, \pm k, -k, \pm k, \pm k, -k, \dots)$ of length $k-2$. In the next lemma we claim that X_1^k really describes the state after the first stage of computation, where input is a balanced 2-flat sequence.

Lemma 6. Let $k \geq 3$ and let $\bar{c} = (c_1, \dots, c_{b_k})$ be a balanced 2-flat sequence of integers from $[0, 2^{k-1} - 1]$. Let $s = \text{height}(\bar{c})$ and let $\bar{d} = \text{reduce}(\bar{c})$. Then $(\otimes \hat{Q}_1^k)(\bar{d}) \in I(X_1^k)$.

Proof. Recall that $H_i = 2^i - 1$. Let $\bar{d} = (d_1, \dots, d_{k-2})$. By Definitions 5 and 6 $s = c_i + c_{b_k-i+1}$ and each $d_i = c_i - \frac{s}{2} = \frac{c_i - c_{b_k-i+1}}{2}$. Observe that each $d_i \in I(\pm k) = [-\frac{H_{k-1}}{2}, \frac{H_{k-1}}{2}]$. It follows from the following sequence of inequalities: $-\frac{H_{k-1}}{2} \leq \frac{-c_{b_k-i+1}}{2} \leq \frac{c_i - c_{b_k-i+1}}{2} \leq \frac{c_i}{2} \leq \frac{H_{k-1}}{2}$. Moreover, the sequence \bar{d} is 2-flat, because \bar{c} is 2-flat. That means that $d_1 \leq d_3 \leq d_5 \leq \dots \leq d_{k'} \leq d_1 + 1$ and $d_2 \leq d_4 \leq d_6 \leq \dots \leq d_{k''} \leq d_2 + 1$, where $k' = 2\lceil \frac{k-2}{2} \rceil - 1$ and $k'' = 2\lfloor \frac{k-2}{2} \rfloor$.

Fact 12 Either $-\frac{1}{2} \leq d_1$ and $d_{k''} \leq 0$ or $-\frac{1}{2} \leq d_2$ and $d_{k'} \leq 0$.

To prove the fact we consider three cases of the value of d_1 .

Case $d_1 \geq 0$: In this case we only have to prove that $d_{k''} \leq 0$. But it is true since $d_{k''} = \frac{c_{k''} - c_{b_k-k''+1}}{2} \leq \frac{c_{b_k} - c_1}{2} = -d_1 \leq 0$. The last inequality holds, because \bar{c} is 2-flat and both k'' and b_k are even.

Case $d_1 \leq -1$: Then $d_{k'} \leq d_1 + 1 \leq 0$. Thus we have only to prove that $d_2 \geq -\frac{1}{2}$. Similar to the previous case, we observe that $d_2 = \frac{c_2 - c_{b_k-1}}{2} \geq \frac{c_{b_k} - 1 - (c_1 + 1)}{2} = -d_1 - 1 \geq 0$.

Case $d_1 = -\frac{1}{2}$: Then $d_{k'} \leq d_1 + 1 = \frac{1}{2}$ and from $-\frac{1}{2} = \frac{c_1 - c_{b_k}}{2}$ we get $c_1 + 1 = c_{b_k} \leq c_2 + 1$. Since $c_2 \geq c_1$, we have $d_2 \geq d_1 = -\frac{1}{2}$. If $d_{k'} \leq 0$, we are done. Otherwise $d_{k'} = \frac{1}{2}$ and we have to show that $d_{k''} \leq 0$. To this end let us notice that $\frac{s}{2} = c_1 - d_1 = c_1 + \frac{1}{2}$ and $c_{b_k-k'+1} = s - c_{k'} = s - (d_{k'} + \frac{s}{2}) = \frac{s}{2} - \frac{1}{2} = c_1$. It follows that $c_{k''} = c_1$ since $c_1 \leq c_2 \leq c_{k''} \leq c_{b_k-k'+1} = c_1$. Thus $d_{k''} = d_1 = -\frac{1}{2}$ and this concludes the proof of Fact 12.

From Fact 12 and since \bar{d} is 2-flat we can immediately get the following corollary.

Corollary 3. $\bar{d} \in I((k-1, -k, k-1, -k, \dots) \cup I(-k, k-1, -k, k-1, \dots))$.

To finish the proof of the lemma we need one more fact:

Fact 13 $(\otimes \hat{Q}_1^k)(I((k-1, -k, k-1, -k, \dots) \cup I(-k, k-1, -k, k-1, \dots))) \subseteq I(X_1^k)$.

To prove this fact let us firstly represent X_1^k in the same form as \hat{Q}_1^k is.

$$X_1^k = (k-1) \oplus \bigoplus_{i=1}^{\lfloor \frac{k-3}{2} \rfloor} (\pm k, -k, \pm k) \oplus (Y_1^k),$$

where Y_1^k is empty if $k \equiv 0 \pmod{3}$, $Y_1^k = (\pm k)$ if $k \equiv 1 \pmod{3}$ and $Y_1^k = (\pm k, -k)$ if $k \equiv 2 \pmod{3}$. Looking now at both representations we can see that the output of Cyc function should be in $I(k-1)$, the output of each Dec_i function should be in $I(\pm k)$ and the output of $MinMax$ should be in $I(-k) \times I(\pm k)$. If Min function is used, then its output should be in $I(-k)$. The input to Cyc is either from $I(k-1)$ or from $I(-k)$. In both cases we get desired output according to Fact 11.2. In the similar way, the input to each Dec_i function is either from $I(k-1) \subseteq I(\pm k)$ or from $I(-k) \subseteq I(\pm k)$. But $Dec_i(I(\pm k)) \subseteq I(\pm k)$ by Fact 11.1. From Fact 11.3 we have $Min(I(\pm k)) \subseteq I(-k)$. Finally, the input to $MinMax$ function is either from $I(k-1) \times I(-k)$ or from $I(-k) \times I(k-1)$. For this function the result follows from Fact 11.4. \square

Lemma 7. For $k \geq 3$ and each $i = 1, 2, \dots, 5k-13$ the following inclusion holds:

$$(\bigotimes_{i \bmod 3+1} \hat{Q}_i^k)(I(X_i^k)) \subseteq I(X_{i+1}^k).$$

Proof. We have to prove that for $k \geq 3$ and $x = 1, 2, 3$ the following inclusions are true: $(\bigotimes \hat{Q}_x^k)(I(X_{3j+x-1}^k)) \subseteq I(X_{3j+x}^k)$, where $j = 1, 2, \dots, \lfloor \frac{5k-13}{3} \rfloor$ for $x = 1$ and $j = 0, 1, 2, \dots, \lfloor \frac{5k-12-x}{3} \rfloor$ for $x = 2, 3$. The sequences \hat{Q}_x^k , $x = 1, 2, 3$, are built of functions Cyc , Dec_* , $MinMax$ and Min introduced in Definition 7. We consider these function one after another analysing which positions in state sequences are modified by them and what values are in that positions before and after applying a function. In the following, we denote by $A_{i,j}$ the j -th element of a sequence A_i .

The function Cyc is used only in the definition of \hat{Q}_1^k and is applied to position 1 of state sequences $I(X_{3j}^k)$, where $j = 1, 2, \dots, \lfloor \frac{5k-13}{3} \rfloor$. Thus it is enough to show the inclusion $Cyc(I(X_{3j,1}^k)) \subseteq I(X_{3j+1,1}^k)$. By Definition 11 the argument of $Cyc \cdot I$ can be: $X_{3j,1}^k = V_{0,1}^k = -k$ for $3j \leq 3k-9$ or $X_{3j,1}^k = W_{0,1}^k = 0$ for $3j = 3k-6$ or $X_{3j,1}^k = Z_1 = 0$ for $3j > 3k-6$. The corresponding value of the next state sequence is $X_{3j+1,1}^k = V_{1,1}^k = k-1$ for $3j+1 \leq 3k-8$ or $X_{3j+1,1}^k = Z_1 = 0$ for $3j+1 \geq 3k-5$. Using Fact 11, inclusions $Cyc(I(-k)) \subseteq I(k-1)$ and $Cyc(I(0)) \subseteq I(0)$ are true and we are done.

In the sequence \hat{Q}_1^k we have several Dec_{k-3l} functions, each Dec_{k-3l} is on the corresponding position $3l-1$ and it is applied to the state sequence $I(X_{3j}^k)$, where $l = 1, \dots, \lfloor \frac{k-1}{3} \rfloor$. Similarly, in \hat{Q}_2^k we have several Dec_{k-3l+1} functions, each Dec_{k-3l+1} is on the corresponding position $3l-2$ and it is applied to the state sequence $I(X_{3j+1}^k)$, where $l = 1, \dots, \lfloor \frac{k}{3} \rfloor$. Finally, in \hat{Q}_3^k we have Dec_{k-3l-1} functions, each Dec_{k-3l-1} is on the corresponding position $3l$ and it is applied to the state sequence $I(X_{3j+2}^k)$, where $l = 1, \dots, \lfloor \frac{k-2}{3} \rfloor$. Assuming that \hat{Q}_0^k also denotes \hat{Q}_3^k , we can rewrite our proof goal for that functions as the following fact.

Fact 14 For $k \geq 3$ and $x = 0, 1, 2$ the set $Dec_{k-3l+x-1}(I(X_{3j+x-1,3l-x}^k))$ is a subset of $I(X_{3j+x,3l-x}^k)$, where $l = 1, \dots, \lfloor \frac{k-2+x}{3} \rfloor$, $j = 1, 2, \dots, \lfloor \frac{5k-12-x}{3} \rfloor$ for $x = 0, 1$ and $j = 0, 1, 2, \dots, \lfloor \frac{5k-14}{3} \rfloor$ for $x = 2$.

The sequences X_*^k are defined with the help of sequences U_*^k , V_*^k , W_*^k and Z_* , therefore we prove the fact by considering all possible cases in the following table. In it we assume that $U_{-1}^k = U_2^k$, $V_{-1}^k = V_2^k$ and $W_{-1}^k = W_2^k$.

Cases of $s = X_{3j+x-1, 3l-x}^k$	Cases of $t = X_{3j+x, 3l-x}^k$	Value of s	Value of t	Why $Dec_{k-3l+x-1}(I(s)) \subseteq I(t)?$
$s = U_{x-1, 3l-x}^k$	$t = U_{x, 3l-x}^k$	$\pm k$	$\pm k$	Fact 11.1
$s = V_{x-1, 3l-x}^k$	$t = V_{x, 3l-x}^k$	$k - 3l + x$	$k - 3l + x - 1$	
$s = V_{x-1, 3l-x}^k$	$t = W_{x, 3l-x}^k$	$k - 3l + x$	$k - 3l + x - 1$	
$s = W_{x-1, 3l-x}^k$	$t = W_{x, 3l-x}^k$	$k - 3l + x$	$k - 3l + x - 1$	
$s = Z_{3l-x}$	$t = Z_{3l-x}$	0	0	

The two remaining cases: (1) $X_{3j+x-1, 3l-x}^k = U_{x-1, 3l-x}^k$ and $X_{3j+x, 3l-x}^k = V_{x, 3l-x}^k$ and (2) $X_{3j+x-1, 3l-x}^k = W_{x-1, 3l-x}^k$ and $X_{3j+x, 3l-x}^k = Z_{3l-x}$ are not possible, because, otherwise, (1) $3j+x$ should be even and $\frac{3j+x}{2} = 3l-x-1$, which cannot hold for any integers j , x and l ; (2) $3j+x-(3k-6)$ should be even and $\frac{3j+x-(3k-6)}{2} = 3l-x-1$, which is not true for the same reason.

Now we consider the *Min* function. It appears in the definition of \hat{Q}_1^k (\hat{Q}_2^k or \hat{Q}_3^k , respectively) on the position $k-2$ if $k \bmod 3 = 2$ ($k \bmod 3 = 1$ or $k \bmod 3 = 0$, respectively). Thus, to prove the lemma, it suffices to show the following fact.

Fact 15 For $k \geq 3$ and $x = 0, 1, 2$ the set $Min(I(X_{3j+x-1, k-2}^k))$ is a subset of $I(X_{3j+x, k-2}^k)$, where $k-2 \equiv 1-x \pmod{3}$ $j = 1, 2, \dots, \lfloor \frac{5k-12-x}{3} \rfloor$ for $x = 0, 1$ and $j = 0, 1, 2, \dots, \lfloor \frac{5k-14}{3} \rfloor$ for $x = 2$.

As in the case of Dec_* functions we prove the fact by considering all possible cases in the following table. In it we assume that $U_{-1}^k = U_2^k$, $V_{-1}^k = V_2^k$ and $W_{-1}^k = W_2^k$.

Cases of $s = X_{3j+x-1, k-2}^k$	Cases of $t = X_{3j+x, k-2}^k$	Value of s	Value of t	Why $Min(I(s)) \subseteq I(t)?$
$s = U_{x-1, k-2}^k$	$t = U_{x, k-2}^k$	$\pm k$	$-k$	Fact 11.3
	$t = V_{x, k-2}^k$	$\pm k$	$-k$	
$s = V_{x-1, k-2}^k$	$t = W_{x, k-2}^k$	1	0	
$s = W_{x-1, k-2}^k$	$t = W_{x, k-2}^k$	1	0	
	$t = Z_{x, k-2}$	1	0	
$s = Z_{k-2}$	$t = Z_{k-2}$	0	0	

The remaining case $X_{3j+x-1, k-2}^k = V_{x-1, k-2}^k$ and $X_{3j+x, k-2}^k = V_{x, k-2}^k$ is not possible, because, otherwise $3j+x-1 = 2k-5$, that is, $2(k-2) = 3j+x$, but $k-2 \equiv 1-x \pmod{3}$ and in the consequence $x \equiv 2(1-x) \pmod{3}$ - contradiction.

The last function we have to consider is *MinMax*, which appears in the definition of all \hat{Q}_x^k , $x = 1, 2, 3$, functions. In \hat{Q}_1^k (\hat{Q}_2^k and \hat{Q}_3^k , respectively) a copy of *MinMax* is on positions $(3, 4)$, $(6, 7)$, \dots $((2, 3)$, $(5, 6)$, \dots and $(1, 2)$, $(4, 5)$, \dots , respectively). Thus, to prove the lemma, it suffices to show the following fact.

Fact 16 For $k \geq 3$ and $x = 1, 2, 3$ the set $MinMax(I(X_{3j+x-1, 3l-x+1}^k, X_{3j+x-1, 3l-x+2}^k))$ is a subset of $I(X_{3j+x, 3l-x+1}^k, X_{3j+x, 3l-x+2}^k)$, where $l = 1, \dots, \lfloor \frac{k-4+x}{3} \rfloor$ $j = 1, 2, \dots, \lfloor \frac{5k-13}{3} \rfloor$ for $x = 1$ and $j = 0, 1, 2, \dots, \lfloor \frac{5k-12-x}{3} \rfloor$ for $x = 2, 3$.

As in the case of previous functions we prove the fact by considering all possible cases in the following table. In it we assume that $U_3^k = U_0^k$, $V_3^k = V_0^k$ and $W_3^k = W_0^k$. To reduce the size of the table we also use the following shortcuts: $a = 3j + x$, $b = 3l - x + 1$ and $y = k - 3l + x - 2$. Observe that $2 \leq y \leq k - 2$, therefore $I(0) \subseteq I(y) \subseteq I(\pm k)$ and we can also apply Fact 11.5.

Cases of (s_1, s_2)		Cases of (t_1, t_2)		Value of		Value of		Why $I(t_1, t_2) \supseteq \text{MinMax}(I(s_1, s_2))$?
$X_{a-1, b-1}^k$	$X_{a-1, b}^k$	$X_{a, b-1}^k$	$X_{a, b}^k$	s_1	s_2	t_1	t_2	
$U_{x-1, b-1}^k$	$U_{x-1, b}^k$	$U_{x, b-1}^k$	$U_{x, b}^k$	$\pm k$	$-k$	$-k$	$\pm k$	Fact 11.4
		$V_{x, b-1}^k$	$U_{x, b}^k$	$\pm k$	$-k$	$-k$	$\pm k$	
$V_{x-1, b-1}^k$	$U_{x-1, b}^k$	$V_{x, b-1}^k$	$U_{x, b}^k$	y	$-k$	$-k$	$\pm k$	Fact 11.5
			$V_{x, b}^k$	y	$-k$	$-k$	y	
	$V_{x-1, b}^k$	$V_{x, b-1}^k$	$V_{x, b}^k$	y	$-k$	$-k$	y	
			$W_{x, b}^k$	y	$-k$	$-k$	y	
$W_{x-1, b-1}^k$	$W_{x-1, b}^k$	$W_{x, b-1}^k$	$W_{x, b}^k$	y	0	0	y	
			$W_{x, b}^k$	y	0	0	y	
		Z_{b-1}	$W_{x, b}^k$	0	0	0	y	
Z_{b-1}	$W_{x-1, b}^k$	Z_{b-1}	$W_{x, b}^k$	0	0	0	0	
	Z_b	Z_{b-1}	Z_b	0	0	0	0	

□

Lemma 8. Let $k \geq 3$ and let $\bar{c} = (c_1, \dots, c_{b_k})$ be a balanced 2-flat sequence of integers from $[0, 2^{k-1} - 1]$ and let $s = \text{height}(\bar{c})$. Let $f = f_{5k-12} \circ f_{5k-13} \circ \dots \circ f_1$, where $f_i = Q_{((i-1) \bmod 3)+1}^k$, $i = 1, \dots, 5k - 12$. Then $f(\bar{c}) = (\frac{s}{2})^{b_k}$ if s is even or $f(\bar{c}) = (\frac{s-1}{2})^{k-2} \oplus (\frac{s+1}{2})^{k-2}$ otherwise.

Proof. Since each f_i maps a balanced sequence to a balanced one, let $\hat{f}_i = \text{reduce}(f_i, s) = \otimes \hat{Q}_{((i-1) \bmod 3)+1}^k$, where the later equality follows from Lemma 5. Let also $\bar{d}_0 = \text{reduce}(\bar{c})$ and let $\bar{d}_i = \hat{f}_i(\bar{d}_{i-1})$ for $i = 1, \dots, 5k - 12$. Then $\bar{d}_1 \in I(X_1^k)$ by Lemma 6 and for $i = 2, \dots, 5k - 12$ we get $\bar{d}_i \in I(X_i^k)$ by an easy induction and Lemma 7. Let \mathbb{Z} denote as usual the set of integers. By $\mathbb{Z}_{\frac{1}{2}}$ we will denote the set $\{z + \frac{1}{2} | z \in \mathbb{Z}\}$. Looking at Definitions 6 and 7 observe the following fact:

Fact 17 If s is even then all elements of sequences \bar{d}_i , $i = 0, \dots, 5k - 12$, are integers. If s is odd then all elements of sequences \bar{d}_i , $i = 0, \dots, 5k - 12$, are in $\mathbb{Z}_{\frac{1}{2}}$.

Since $\bar{d}_{5k-12} \in I(X_{5k-12}^k) = I(0^{k-2})$ and $I(0) \cap \mathbb{Z} = \{0\}$ and $I(0) \cap \mathbb{Z}_{\frac{1}{2}} = \{\frac{1}{2}\}$, it follows that $\bar{d}_{5k-12} = 0^{k-2}$ if s is even and $\bar{d}_{5k-12} = \frac{1}{2}^{k-2}$, otherwise. Applying now the definition of s -extended sequence to 0^{k-2} and $\frac{1}{2}^{k-2}$ we get the desired conclusion of the lemma. □

In this way, with respect to Lemma 3, we have proved that the network M_k is able to merge in $5k - 12$ stages two sorted sequences given in odd and even registers, provided that the numbers of ones in our matrix columns form a balanced sequence. If the sequence is not balanced, $k - 3$ additional stages are needed to get a sorted output.

3.3 Analysis of General Columns

In a general case we will use balanced sequences as lower and upper bounds on the numbers of ones in our matrix columns and observe that Q_1^k , Q_2^k and Q_3^k are monotone functions (see Fact 9).

Definition 12. Let $k \geq 3$ and let $\bar{c} = (c_1, \dots, c_{b_k})$ be a 2-flat sequence of integers from $[0, 2^{k-1} - 1]$ that is not balanced. Since both $\bar{c}_{odd} = (c_1, \dots, c_{b_k-1})$ and $\bar{c}_{evn} = (c_2, \dots, c_{b_k})$ are flat sequences, let i (j , respectively) be such that $c_{2i-1} < c_{2i+1}$ ($c_{b_k-2j} < c_{b_k-2j+2}$, respectively) or let $i = k-2$ ($j = k-2$) if \bar{c}_{odd} (\bar{c}_{evn} , respectively) is a constant sequence. The defined below sequences \check{c} and \hat{c} we will call lower and upper bounds of \bar{c} . If $i < j$ then for $l = 1, \dots, b_k$

$$\check{c}_l = \begin{cases} c_1 & \text{if } l \text{ is odd and } l \leq 2j-1 \\ c_{b_k-1} & \text{if } l \text{ is odd and } l \geq 2j+1 \\ c_l & \text{if } l \text{ is even} \end{cases} \quad \hat{c}_l = \begin{cases} c_{b_k-1} & \text{if } l \text{ is odd} \\ c_{b_k} & \text{if } l \text{ is even} \end{cases}$$

If $i > j$ then for $l = 1, \dots, b_k$

$$\check{c}_l = \begin{cases} c_1 & \text{if } l \text{ is odd} \\ c_2 & \text{if } l \text{ is even} \end{cases} \quad \hat{c}_l = \begin{cases} c_l & \text{if } l \text{ is odd} \\ c_2 & \text{if } l \text{ is even and } l \leq b_k - 2i \\ c_{b_k} & \text{if } l \text{ is even and } l > b_k - 2i \end{cases}$$

Fact 18 For $k \geq 3$ and any not balanced 2-flat sequence $\bar{c} = (c_1, \dots, c_{b_k})$ of integers from $[0, 2^{k-1} - 1]$ the sequences \check{c} and \hat{c} are balanced, $\text{height}(\check{c}) + 1 = \text{height}(\hat{c})$ and $\check{c} \leq \bar{c} \leq \hat{c}$.

Proof. Let i and j be defined as in Definition 12. We will only consider the case $i < j$. The proof of the other case is similar. Directly from the definition we get that \hat{c} is balanced. To see that \check{c} is also balanced let us check for $l = 1, \dots, k-2$ whether the sum $\check{c}_{2l-1} + \check{c}_{b_k-2l+2}$ is constant.

$$\check{c}_{2l-1} + \check{c}_{b_k-2l+2} = \check{c}_{2l-1} + c_{b_k-2l+2} = \begin{cases} c_1 + c_{b_k-2l+2} = c_1 + c_{b_k} & \text{if } l \leq j \\ c_{b_k-1} + c_{b_k-2l+2} = c_{b_k-1} + c_2 & \text{otherwise} \end{cases}$$

If $j = k-2$ there is no otherwise case and we are done. If $j < k-2$ then $c_{b_k} - c_2 = c_{b_k-1} - c_1 = 1$, because of the definition of i and j and we are also done. Moreover $\text{height}(\check{c}) + 1 = c_1 + c_{b_k} + 1 = c_{b_k-1} + c_{b_k} = \text{height}(\hat{c})$. To prove that $\check{c} \leq \bar{c} \leq \hat{c}$ we consider even and odd indices. For even indices from the definition we have: $\check{c}_{2l} = c_{2l} \leq c_{b_k} = \hat{c}_{2l}$. For odd indices $\hat{c}_{2l-1} = c_{b_k-1} \geq c_{2l-1} \geq c_1$. If $l \leq j$ we are done, otherwise, $c_{2l-1} = c_{b_k-1} = \check{c}_{2l-1}$, because \bar{c}_{odd} is flat. \square

Theorem 19. Let $k \geq 3$ and let $\bar{c} = (c_1, \dots, c_{b_k})$ be a 2-flat sequence of integers from $[0, 2^{k-1} - 1]$. Let $f = f_{6k-15} \circ f_{6k-14} \circ \dots \circ f_1$, where $f_i = Q_{((i-1) \bmod 3)+1}^k$, $i = 1, \dots, 6k-15$. Then $f(\bar{c})$ is a flat sequence.

Proof. For a 2-flat sequence \bar{c} of integers from $[0, 2^{k-1} - 1]$ let \check{c} and \hat{c} be its balanced lower and upper bounds, as defined in Definition 12. Let $\bar{c}_0 = \bar{c}$, $\check{c}_0 = \check{c}$, $\hat{c}_0 = \hat{c}$ and for $i = 1, \dots, 6k-15$ let us define $\bar{c}_i = f_i(\bar{c}_{i-1})$, $\check{c}_i = f_i(\check{c}_{i-1})$ and $\hat{c}_i = f_i(\hat{c}_{i-1})$. Observe that $\check{c}_i \leq \bar{c}_i \leq \hat{c}_i$, because of monotonicity of functions Q_1^k , Q_2^k , Q_3^k and Fact 18. To prove that \bar{c}_{6k-15} is a flat sequence we need the following three technical facts.

Fact 20 Let $s = \text{height}(\check{c})$. If s is even then $\bar{c}_{i,j} = \frac{s}{2}$ and $\bar{c}_{i,b_k-j+1} \in \{\frac{s}{2}, \frac{s}{2} + 1\}$ for each $i = 3k - 6, \dots, 5k - 12$ and $j = 1, \dots, \lceil \frac{i+1-(3k-6)}{2} \rceil$. If s is odd then $\bar{c}_{i,j} \in \{\frac{s-1}{2}, \frac{s+1}{2}\}$ and $\bar{c}_{i,b_k-j+1} = \frac{s+1}{2}$ for each $i = 3k - 6, \dots, 5k - 12$ and $j = 1, \dots, \lceil \frac{i+1-(3k-6)}{2} \rceil$.

Proof. Since both \check{c} and \hat{c} are balanced, we can consider reduced forms of them and use Lemmas 6 and 7. For the given range of i 's values that means that

$$\text{reduce}(\check{c}_i), \text{reduce}(\hat{c}_i) \in I(X_i^k) = I(\text{join}_k(\lceil \frac{i+1-(3k-6)}{2} \rceil, Z^k, W_i^k)).$$

It follows that for a given range of j 's values $\text{reduce}(\check{c}_i)_j, \text{reduce}(\hat{c}_i)_j \in I(0) = [-\frac{1}{2}, 0]$. From Fact 18 we know that $\text{height}(\hat{c}) = s + 1$ and from Lemma 4 that heights are pre-served in sequences \check{c}_i and \hat{c}_i . Thus, from the definition of a reduced sequence, $\check{c}_{i,j} \in [\frac{s-1}{2}, \frac{s}{2}]$, $\check{c}_{i,b_k-j+1} \in [\frac{s}{2}, \frac{s+1}{2}]$, $\hat{c}_{i,j} \in [\frac{s}{2}, \frac{s+1}{2}]$ and $\hat{c}_{i,b_k-j+1} \in [\frac{s+1}{2}, \frac{s+2}{2}]$. Since \check{c}_i and \hat{c}_i are sequences of integers, for even s we get $\check{c}_{i,j} = \check{c}_{i,b_k-j+1} = \hat{c}_{i,j} = \frac{s}{2}$ and $\hat{c}_{i,b_k-j+1} = \frac{s+2}{2}$; for odd s we conclude that $\check{c}_{i,j} = \frac{s-1}{2}$ and $\check{c}_{i,b_k-j+1} = \hat{c}_{i,j} = \hat{c}_{i,b_k-j+1} = \frac{s+1}{2}$. Since $\check{c}_{i,j} \leq \bar{c}_{i,j} \leq \hat{c}_{i,j}$, the fact follows. \square

The second fact extends the first fact up to the last stage of our computation.

Fact 21 Let $s = \text{height}(\check{c})$. If s is even then $\bar{c}_{i,j} = \frac{s}{2}$ and $\bar{c}_{i,b_k-j+1} \in \{\frac{s}{2}, \frac{s}{2} + 1\}$ for each $i = 5k - 11, \dots, 6k - 15$ and $j = 1, \dots, k - 2$. If s is odd then $\bar{c}_{i,j} \in \{\frac{s-1}{2}, \frac{s+1}{2}\}$ and $\bar{c}_{i,b_k-j+1} = \frac{s+1}{2}$ for each $i = 5k - 11, \dots, 6k - 15$ and $j = 1, \dots, k - 2$.

Proof. Consider first the sequence \bar{c}_{5k-12} and observe that for $i = 5k - 12$ the value of $\lceil \frac{i+1-(3k-6)}{2} \rceil$ is equal to $k - 2$. It follows from Fact 20 that for even s all values from the left half of \bar{c}_{5k-12} are equal to $\frac{s}{2}$ and all values from the right half of \bar{c}_{5k-12} are in $\{\frac{s}{2}, \frac{s}{2} + 1\}$. For odd s all values from the left half of \bar{c}_{5k-12} are in $\{\frac{s-1}{2}, \frac{s+1}{2}\}$ and all values from the right half of \bar{c}_{5k-12} are equal to $\frac{s+1}{2}$. Since Q_1^k, Q_2^k and Q_3^k are built of functions $\text{dec}_*^k, \text{mov}_*^k$ and cyc^k (cf. Definitions 3 and 4) observe that each function f_i , $i = 5k - 11, \dots, 6k - 15$ can only exchange values at positions from $\text{args}(\text{mov}_*^k)$ that are from non-constant half of arguments (in case of dec_*^k and cyc^k we can observe that for $a \leq b \leq a + 1$ and any $h \geq 0$ we have $\min(a, b + h) = a$, $\max(a - h, b) = b$, $\max(a, b - 1) = a$ and $\min(a + 1, b) = b$, that is, the functions are identity mappings in stages $5k - 11, \dots, 6k - 15$). The mov_*^k functions can only exchange unequal values at neighbour positions moving the smaller value to the left. \square

The last fact states that unequal values $\bar{c}_{i,j}$ described in the previous two facts are getting sorted during the computation. Observe that if s is odd (even, respectively) then we only have to trace the sorting process in a left (right, respectively) region of indices $[1, \min(k - 2, \lceil \frac{i+1-(3k-6)}{2} \rceil)]$ ($[\max(k - 1, b_k - \lceil \frac{i+1-(3k-6)}{2} \rceil + 1), b_k]$, respectively), where $i = 3k - 6, \dots, 6k - 15$ and the values to be sorted differs at most by one. We trace the positions of the smaller values $s' = \frac{s-1}{2}$ in the left region and the greater values $s' = \frac{s}{2} + 1$ in the right region. We will call s' a moving element. For $t = 1, \dots, k - 2$ let us define $i_t = 3k + 2t - 8$ to be the stage, after which the length of the region extends from $t - 1$ to t and a new element appears in it. Let $t' = t$ for odd s and $t' = b_k - t + 1$,

otherwise, be the position of this new element and $a_t = c_{i_t, t'}$ be its value. Finally, let $n_t = |\{1 \leq l \leq t \mid a_l = s'\}|$ be the number of moving elements in the region after stage i_t .

Fact 22 *Using the above definitions, for $t = 1, \dots, k-2$, if $a_t = s'$ then for $i = 0, \dots, 6k-15-i_t$ we have $c_{i_t+i, \max(t-i, n_t)} = a_t$ if s is odd and $c_{i_t+i, \min(t'+i, b_k-n_k+1)} = a_t$, otherwise.*

Proof. We prove the fact only for odd s , that is, for the left region. The proof for the right region is symmetric. We would like to show that if $a_t = s'$ appears at position $t' = t$ after stage i_t then it moves in each of the following stages one position to the left up to its final position n_t . The proof is by induction on t and i . If $t = 1$ and $a_1 = s'$ appears at position 1 after stage $i_1 = 3k-6$ then $n_1 = 1$ and a_1 is already at its final position. It never moves, because values at second position are $\geq s'$, by Facts 20 and 21. If $t > 1$ and $a_t = s'$ then the basis $i = 0$ is obviously true. In the inductive step $i > 0$ we assume that $c_{i_t+i-1, \max(t'-i+1, n_t)} = a_t$ and that the fact is true for smaller values of t . If $\max(t-i+1, n_t) = n_t$ then also $\max(t-i, n_t) = n_t$ and, by the induction hypothesis, values at positions $1, \dots, n_t-1$ are all equal s' . That means that a_t is at its final position and we are done. Thus we left with the case: $n_t < t-i+1$, that is, with $n_t \leq t-i$.

Consider the sequences \bar{c}_{i_t+i-1} and $\bar{c}_{i_t+i} = f_{i_t+i}(\bar{c}_{i_t+i-1})$. We know that $\bar{c}_{i_t+i-1, t-i+1} = s'$. To prove that $\bar{c}_{i_t+i, t-i} = s'$ we would like to show that $\bar{c}_{i_t+i-1, t-i} = s' + 1$ and $\text{mov}_{t-i}^k \in f_{i_t+i}$. The later is a direct consequence of an observation that $\text{mov}_a^k \in f_b$ if and only if $(a+b) \equiv 1 \pmod{3}$. In our case $(t-i) + (i_t+i) = t+i_t = t+3k+2t-8 \equiv 1 \pmod{3}$. To prove the former, let us consider $a_u = s'$, $u \leq t-1$. Then $i_u \leq i_t-2$ and $n_u \leq n_t-1$. By the induction hypothesis, $c_{i_u+j, \max(u-j, n_u)} = s'$. Setting $j = i_t - i_u + i - 1$ we get $j \geq i+1$ and $\max(u-j, n_u) \leq \max(t-1-(i+1), n_t-1) < \max(t-i, n_t) = t-i$. Moreover, $i_u + j = i_t + i - 1$. That means that in the sequence \bar{c}_{i_t+i-1} none of n_t elements s' is at position $t-i$ and, consequently, $\bar{c}_{i_t+i-1, t-i} = s' + 1$. Since mov_{t-i}^k switches s' with $s' + 1$, this completes the proof of Fact 22. \square

Now we are ready to prove that \bar{c}_{6k-15} is a flat sequence. By Fact 21, if s is odd then $\bar{c}_{6k-15} \in \{\frac{s-1}{2}, \frac{s+1}{2}\}^{k-2} (\frac{s+1}{2})^{k-2}$, otherwise, $\bar{c}_{6k-15} \in (\frac{s}{2})^{k-2} \{\frac{s}{2}, \frac{s}{2}+1\}^{k-2}$. The number of minority elements in \bar{c}_{6k-15} has been denote by n_{k-2} . If s is odd and $a_t, t = 1, \dots, k-2$, is a minority element $\frac{s-1}{2}$, then, by Fact 22, $c_{6k-15, n_t} = \frac{s-1}{2}$. If s is even and $a_t, t = 1, \dots, k-2$, is a minority element $\frac{s}{2} + 1$, then, by Fact 22, $c_{6k-15, b_k-n_t+1} = \frac{s}{2} + 1$. In both cases this proves that \bar{c}_{6k-15} is flat, which completes the proof of Theorem 19. \square

3.4 Proof of Theorem 2

Theorem 2 follows directly from Theorem 19 and Lemma 3. Let $k \geq 3$ and \bar{c} be any 2-flat sequence of integers from $[0, 2^{k-1}-1]$. By Theorem 19 the result of application $(Q_3^k \circ Q_2^k \circ Q_1^k)^{2k-5}$ to (\bar{c}) is a flat sequence. Then, by Lemma 3, the network M_k is a $2k-5$ -pass merger of two sorted sequences given in odd and even registers, respectively.

4 Conclusions

For each $k \geq 3$ we have shown a construction of a 3-periodic merging comparator network of $N_k = 2^k(k-2)$ registers and proved that it merge any two sorted sequences

(given in odd and even registers, respectively) in time $6k - 15 = 3(k - 5)$. A natural question remains whether it is the optimal merging time for 3-periodic comparator networks.

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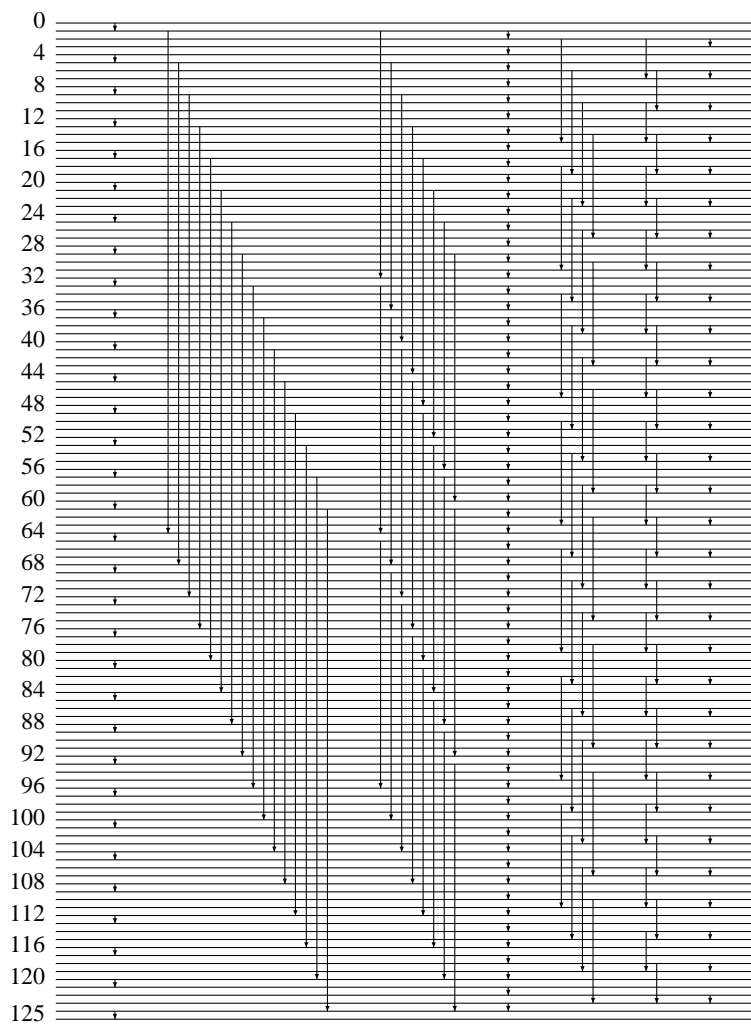


Fig. 6. The P'_6 network